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On Estimates of Initial Coefficients in Some Subclasses of Analytic and Bi-Univalent Functions

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Abstract: **In the present paper, we investigate the issue of obtaining upper bounds for some initial coefficients in some subclasses of analytic and bi-univalent functions, which are defined by Murugusundaramoorthy (2013) and by Frasin (2014). We aim to find an estimate for the fourth coefficient in some of these subclasses. We also aim to estimate the second Hankel determinant for these subclasses.**

Keywords: **Univalent functions, Bi-univalent functions, Starlike functions, Convex function, Fekete-Szegö inequalities, Coefficient bounds, Hankel determinants, Bi-starlike functions, Bi-convex functions.**

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathrm{D}=\{z\in\mathbb{C}:|z|<1\}$ and normalized by $f(0)=f'(0)-1=0$. Such functions may be expressed by the power series

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1.1}$$

We let S denote the subclass of functions in $\mathcal{A}$ that are univalent in D. By the Koebe One-Quarter Theorem, we know that the range of every function in S contains the disk $\{w:|w|<1/4\}$. Therefore, every univalent function f has an inverse f^{-1} so that

$$f^{-1}(f(z))=z\ ,\ (z\in\mathrm{D}) \tag{1.2}$$

Thus,

$$f\left(f^{-1}(w)\right)=w\ ,\ \left(|w|<r_0(f);r_0(f)\geq 1/4\right) \tag{1.3}$$

In fact, the inverse function $g=f^{-1}$ is given by the power series

$$g(w)=f^{-1}(w)=w+\sum_{n=2}^{\infty}b_n w^n=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\ldots \tag{1.4}$$

A function $f\in\mathcal{A}$ is said to be bi-univalent in D if both f and its inverse $g=f^{-1}$ map are univalent in D. Let σ (or Σ) be the class of all bi-univalent functions in D having the series expansion (1.1). Examples of functions in the class σ are

$$\frac{z}{1-z} \quad , \quad -\log(1-z) \quad , \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and so on.

There are other common examples of functions in S that are not members of σ, such as

$$z-\frac{1}{2}z^2 \quad , \quad \frac{z}{1-z^2} \quad , \quad \frac{z}{(1-z)^2}$$

The research into σ was started by Lewin ([1], 1967). It focused on problems related to coefficients. Many papers concerning bi-univalent functions have been published recently, among others, by Srivastava et al ([2], 2010) and Frasin and Aouf ([3], 2011). Hamidi and Jahangiri ([4], 2014) have revealed the importance of Faber polynomials in studying coefficients of bi-univalent functions.

A function $f(z)$ belonging to S is said to be starlike of order α if it satisfies

$$\mathrm{R}\left(\frac{zf'(z)}{f(z)}\right)>\alpha \quad ,(z\in \mathrm{D}) \ , \ (0\le\alpha<1) \tag{1.5}$$

We denote the subclass of S consisting of functions which are starlike of order α in D by $S^*(\alpha)$. For $\alpha=0$, we get the subclass of starlike functions which is denoted by $S^*=S^*(0)$

A function $f(z)$ belonging to S is said to be convex of order α if it satisfies

$$\mathrm{R}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \quad ,(z\in \mathrm{D}) \ , \ (0\le\alpha<1) \tag{1.6}$$

We denote the subclass of S consisting of functions which are convex of order α in D by $\mathcal{K}(\alpha)$. For $\alpha=0$, we get the subclass of convex functions which is denoted by $\mathcal{K}=\mathcal{K}(0)$.

For $(0\le\beta<1)$, a function $f\in\sigma$ is in the subclass $S_\sigma^*(\beta)$ of bi-starlike functions of order β if both f and its inverse map f^{-1} are starlike of order β. For $\beta=0$, we get the subclass of bi-starlike functions which is denoted by $S_\sigma^*=S_\sigma^*(0)$.

For $(0\le\beta<1)$, a function $f\in\sigma$ is in the subclass $\mathcal{K}_\sigma(\beta)$ of bi-convex functions of order β if both f and its inverse map f^{-1} are convex of order β. For $\beta=0$, we get the subclass of bi-convex functions which is denoted by $\mathcal{K}_\sigma=\mathcal{K}_\sigma(0)$.

The class $\mathcal{P}$ denotes all functions h analytic and having positive real part in D, with $h(0)=1$. Such functions may be expressed by the power series

$$h(z)=1+\sum_{k=2}^{\infty}b_k z^k \tag{1.7}$$

In 1967, Lewin [1] showed that for every function $f\in\sigma$ of the form (1.1), the second coefficient satisfies the inequality $|a_2|<1.51$. In the same year also, Brannan and Clunie [5] conjectured that $|a_2|\le\sqrt{2}$ for every $f\in\sigma$. In 1985, Kedzierawski [7] proved Brannan and Clunie's conjecture for bi-starlike functions $f\in S_\sigma^*$. Brannan and Taha [8] obtained the estimates on the initial coefficients $|a_2|$ and $|a_3|$ for functions in the subclasses $S_\sigma^*(\beta)$ and $\mathcal{K}_\sigma(\beta)$.

A lot of results for $|a_2|$, $|a_3|$ and $|a_4|$ were proved for some subclasses of σ. However, they are not sharp. The problem of estimating coefficients $|a_n| \le n$, $n \ge 2$ is still open, though.

2. BASIC DEFINITIONS AND SOME NOTATIONS

One of the important tools in the theory of univalent functions is Hankel determinants [9]. The Hankel determinants $H_q(n)$, $(n=1,2,...,\ q=1,2...)$ of the function f are defined by

$$H_q(n)=\begin{bmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{bmatrix} \qquad (a_1=1) \tag{2.1}$$

This determinant was discussed by several authors with $q=2$. For example, for $n=1$, we get

$$H_2(1)=\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \qquad (a_1=1) \tag{2.2}$$

The functional $H_2(1)=a_3-a_2^2$ is known as the Fekete-Szegö functional when further generalized as $a_3-\mu a_2^2$ where μ is some real number (Duren, [10]). For $n=2$, we get

$$H_2(2)=\begin{bmatrix} a_2 & a_3 \\ a_3 & a_4 \end{bmatrix} \tag{2.3}$$

where the functional $H_2(2)=a_2a_4-a_3^2$ is known as the second Hankel determinant.

In 2013, Murugusundaramoorthy et al. [11] defined two new subclasses $S_\sigma(\alpha,\lambda)$ and $\mathscr{M}_\sigma(\beta,\lambda)$ of σ, which will be defined later; and obtained estimates for coefficients $|a_2|$ and $|a_3|$ in each of these subclasses.

For $f\in\sigma$, let $F(z)$ and $G(w)$ be defined as follows

$$F(z)=\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} \quad , \quad G(w)=\frac{wg'(w)}{(1-\lambda)g(w)+\lambda wg'(w)} \tag{2.4}$$

where $z\in \mathrm{D}$, $g=f^{-1}$, $0\le\lambda<1$.

Definition 2.1

A function $f\in\sigma$ is said to be in the class $S_\sigma(\alpha,\lambda)$ if it satisfies

$$|\arg F(z)|<\frac{\alpha\pi}{2} \quad , \quad |\arg G(w)|<\frac{\alpha\pi}{2} \quad ,(z\in \mathrm{D}) \tag{2.5}$$

where $0<\alpha\le 1$, $0\le\lambda<1$.

Definition 2.2

A function $f\in\sigma$ is said to be in the class $\mathscr{M}_\sigma(\beta,\lambda)$ if it satisfies

$$\mathrm{R}F(z)>\beta \quad , \quad \mathrm{R}G(w)>\beta \quad ,(z\in \mathrm{D}) \tag{2.6}$$

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where $0 \le \beta < 1$, $0 \le \lambda < 1$.

Murugusundaramoorthy [11] proved the two following theorems:

Theorem 2.3

If $f \in S_\sigma(\alpha,\lambda)$, $0 \le \lambda < 1$ and $0 < \alpha \le 1$, then

$$|a_2| \le \frac{2\alpha}{(1-\lambda)\sqrt{1+\alpha}} \quad , \quad |a_3| \le \frac{4\alpha^2}{(1-\lambda)^2} + \frac{\alpha}{1-\lambda} \tag{2.7}$$

Theorem 2.4

If $f \in \mathscr{M}_\sigma(\beta,\lambda)$, $0 \le \lambda < 1$ and $0 \le \beta < 1$, then

$$|a_2| \le \frac{\sqrt{2(1-\beta)}}{1-\lambda} \quad , \quad |a_3| \le \frac{4(1-\beta)^2}{(1-\lambda)^2} + \frac{1-\beta}{1-\lambda} \tag{2.8}$$

In 2014, Zaprawa [12] found Fekete-Szegö inequalities for the subclasses $S_\sigma(\alpha,\lambda)$ and $\mathscr{M}_\sigma(\beta,\lambda)$; and obtained results concerning $|a_3|$ in each of these subclasses, which were better than the results presented in [11], as follows:

Theorem 2.5

If $f \in S_\sigma(\alpha,\lambda)$, $0 \le \lambda < 1$ and $0 < \alpha \le 1$, then

$$|a_3| \le \begin{cases} \dfrac{4\alpha^2}{(1-\lambda)^2(1+\alpha)} & ; \ 4\alpha \ge (1+\alpha)(1-\lambda) \\ \dfrac{\alpha}{1-\lambda} & ; \ 4\alpha \le (1+\alpha)(1-\lambda) \end{cases} \tag{2.9}$$

Remark

The bound in theorem 2.5 is better than the one obtained in theorem 2.3 with respect to $|a_3|$.

Theorem 2.6

If $f \in \mathscr{M}_\sigma(\beta,\lambda)$, $0 \le \lambda < 1$ and $0 \le \beta < 1$, then

$$|a_3| \le \frac{2(1-\beta)}{(1-\lambda)^2} \tag{2.10}$$

Remark

In order to get the subtraction sign of the results of theorem 2.6 and theorem 2.4 with regard to $|a_3|$, it is enough to study the subtraction sign of $4(1-\beta)^2 - 2(1-\beta)$, where it appears to be positive for $0 \le \beta \le 1/2$; and this means that theorem 2.6 gives better results than theorem 2.4 when $0 \le \beta \le 1/2$.

In 2014, Frasin [13] defined two new subclasses $\mathcal{H}_\sigma(\alpha,\beta)$ and $\mathcal{H}_\sigma(\gamma,\beta)$ of σ, as follows:

Definition 2.7

A function $f \in \sigma$ is said to be in the class $\mathcal{H}_\sigma(\alpha,\beta)$ if it satisfies

$$\left|\arg\left(f'(z)+\beta z f''(z)\right)\right|<\frac{\alpha\pi}{2}\ ,\quad \left|\arg\left(g'(w)+\beta z g''(w)\right)\right|<\frac{\alpha\pi}{2}\ ,(z,w\in \mathrm{D}) \tag{2.11}$$

where $g=f^{-1}$, $\beta>0$, $0<\alpha<1$, $2(1-\alpha)\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{\beta m+1}\le 1$.

Definition 2.8

A function $f\in\sigma$ is said to be in the class $\mathcal{H}_\sigma(\gamma,\beta)$ if it satisfies

$$\mathrm{R}\left(f'(z)+\beta z f''(z)\right)>\gamma\ ,\quad \mathrm{R}\left(g'(w)+\beta z g''(w)\right)>\gamma\ ,(z,w\in \mathrm{D}) \tag{2.12}$$

where $g=f^{-1}$, $\beta>0$, $0\le\gamma<1$, $2(1-\alpha)\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{\beta m+1}\le 1$.

Frasin [13] proved the two following theorems:

Theorem 2.9

If $f\in\mathcal{H}_\sigma(\alpha,\beta)$, then

$$|a_2|\le\frac{2\alpha}{\sqrt{2(\alpha+2)+4\beta(\alpha+\beta+2-\alpha\beta)}}\quad ,\quad |a_3|\le\frac{\alpha^2}{(1+\beta)^2}+\frac{2\alpha}{3(1+2\beta)} \tag{2.13}$$

Theorem 2.10

If $f\in\mathcal{H}_\sigma(\gamma,\beta)$, then

$$|a_2|\le\sqrt{\frac{2(1-\gamma)}{3(1+2\beta)}}\quad ,\quad |a_3|\le\frac{(1-\gamma)^2}{(1+\beta)^2}+\frac{2(1-\gamma)}{3(1+2\beta)} \tag{2.14}$$

From theorems 2.9 and 2.10, Frasin [13] obtained the two following corollaries:

Corollary 2.11

If $f\in\mathcal{H}_\sigma(\alpha,1)$, then

$$|a_2|\le\frac{2\alpha}{\sqrt{2(\alpha+2)+12}}\quad ,\quad |a_3|\le\frac{9\alpha^2+8\alpha}{36} \tag{2.15}$$

Corollary 2.12

If $f\in\mathcal{H}_\sigma(\gamma,1)$, then

$$|a_2|\le\frac{1}{3}\sqrt{2(1-\gamma)}\quad ,\quad |a_3|\le\frac{(1-\gamma)(9(1-\gamma)+8)}{36} \tag{2.16}$$

Lemma 2.13

If $h\in\mathcal{P}$ and $h(z)=1+\sum_{k=1}^{\infty}p_k z^k$, then $|p_k|\le 2$. This inequality is sharp for all positive integers k .

Proof of lemma 2.13 exists in many references including Duren ([10], p.41).

Lemma 2.14

The coefficients of each function $f\in S$ of the form (1.1) satisfy $|a_n|\le n$ for $n=2,3,\ldots$

Remark

Lemma 2.14, known as Bieberbach conjecture, was developed by Bieberbach in 1916. It was an open problem which many researchers attempted to prove later on. However, the results that they reached were not comprehensive. It was not until 1985 that Branges [6] was able to prove it.

3. MAIN RESULTS

In the following theorem, we will obtain an estimate for the second Hankel determinant in the subclass $\mathscr{M}_\sigma(\beta,\lambda)$.

Theorem 3.1

If $f \in \mathscr{M}_\sigma(\beta,\lambda)$, $0 \le \lambda < 1$ and $0 \le \beta < 1$, then

$$|H_2(2)| \le \frac{4\sqrt{2(1-\beta)}}{1-\lambda} + \frac{4(1-\beta)^2}{(1-\lambda)^4} \tag{3.1}$$

Proof

Since

$$H_2(2) = \begin{bmatrix} a_2 & a_3 \\ a_3 & a_4 \end{bmatrix}$$

Thus,

$$|H_2(2)| \le |a_2||a_4| + |a_3^2|$$

Hence, according to Bieberbach conjecture and depending on theorem 2.4 and its improvement in theorem 2.6 with regard to coefficient a_3, we obtain the following:

$$|H_2(2)| \le \frac{\sqrt{2(1-\beta)}}{1-\lambda}(4) + \left(\frac{2(1-\beta)}{(1-\lambda)^2}\right)^2 \tag{3.2}$$

Thus, the proof is complete.

Remark

It is worth mentioning that the inequality (3.1) is more precise when $0 \le \beta \le 1/2$.

In the following theorem, we will obtain an estimate for coefficient a_4 in the subclass $\mathcal{H}_\sigma(\gamma,\beta)$.

Theorem 3.2

If $f \in \mathcal{H}_\sigma(\gamma,\beta)$, $0 \le \gamma < 1$ and $\beta > 0$, then

$$|a_4| \le \frac{(1-\gamma)}{2(1+3\beta)} \tag{3.3}$$

Proof

For each $f \in \mathcal{H}_\sigma(\gamma,\beta)$, then (2.12) could be written as follows

$$f'(z) + \beta z f''(z) = \gamma + (1-\gamma)p(z) \tag{3.4}$$

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$$g'(w)+\beta w g''(w)=\gamma+(1-\gamma)q(w) \tag{3.5}$$

where $p(z), q(w) \in \mathcal{P}$, and which are expressed by the power series (1.7), thus

$$\gamma+(1-\gamma)p(z)=1+p_1(1-\gamma)z+p_2(1-\gamma)z^2+p_3(1-\gamma)z^3+\ldots \tag{3.6}$$

$$\gamma+(1-\gamma)q(w)=1+q_1(1-\gamma)w+q_2(1-\gamma)w^2+q_3(1-\gamma)w^3+\ldots \tag{3.7}$$

From (1.1) and (1.4) , we get

$$f'(z)+\beta z f''(z)=1+2a_2(1+\beta)z+3a_3(1+2\beta)z^2+4a_4(1+3\beta)z^3+\ldots \tag{3.8}$$

$$g'(w)+\beta w g''(w)=1-2a_2(1+\beta)w+3(2a_2^2-a_3)(1+2\beta)w^2-4(5a_2^3-5a_2a_3+a_4)(1+3\beta)w^3+\ldots \tag{3.9}$$

From (3.8), (3.4) and (3.6), we get the following equations

$$2a_2(1+\beta)=p_1(1-\gamma) \tag{3.10}$$

$$3a_3(1+2\beta)=p_2(1-\gamma) \tag{3.11}$$

$$4a_4(1+3\beta)=p_3(1-\gamma) \tag{3.12}$$

From (3.9), (3.5) and (3.7), we get the following equations

$$-2a_2(1+\beta)=q_1(1-\gamma) \tag{3.13}$$

$$3(2a_2^2-a_3)(1+2\beta)=q_2(1-\gamma) \tag{3.14}$$

$$-4(5a_2^3-5a_2a_3+a_4)(1+3\beta)=q_3(1-\gamma) \tag{3.15}$$

Summing (3.12) and (3.15), we obtain

$$-4(5a_2^3-5a_2a_3)(1+3\beta)=(p_3+q_3)(1-\gamma) \tag{3.16}$$

$$(5a_2^3-5a_2a_3)=\frac{(p_3+q_3)(1-\gamma)}{-4(1+3\beta)} \tag{3.17}$$

Subtracting (3.15) from (3.12), we obtain

$$8a_4(1+3\beta)+4(5a_2^3-5a_2a_3)(1+3\beta)=(p_3-q_3)(1-\gamma) \tag{3.18}$$

From (3.17) and (3.18), we get

$$a_4=\frac{p_3(1-\gamma)}{4(1+3\beta)} \tag{3.19}$$

Applying lemma 2.13 to (3.19), the proof becomes complete.

Remark

It is clear that $\frac{(1-\gamma)}{2(1+3\beta)}$ is less than one when $\beta>0,\ 0\le\gamma<1$, the thing that highlights the quality of this estimate for the fourth coefficient in the subclass $\mathcal{H}_\sigma(\gamma,\beta)$ in comparison with Bieberbach conjecture.

From theorem 3.2, we get the following:

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Corollary 3.3

If $f \in \mathcal{H}_\sigma(\gamma,1)$, then

$$|a_4| \le \frac{(1-\gamma)}{8} \tag{3.20}$$

Proof

The proof could be directly obtained from (3.3) by considering $\beta = 1$.

Corollary 3.4

If $f \in \mathcal{H}_\sigma(\gamma,1)$, then

$$|H_2(2)| \le \frac{(1-\gamma)^{3/2}}{12\sqrt{2}} + \frac{(9\gamma^2-26\gamma+17)^2}{1296} \tag{3.21}$$

Proof

Since

$$H_2(2) = \begin{bmatrix} a_2 & a_3 \\ a_3 & a_4 \end{bmatrix}$$

Thus,

$$|H_2(2)| \le |a_2||a_4| + |a_3^2|$$

Hence, according to corollary 3.3 and corollary 2.12 with regard to coefficients a_2 , a_3 , a_4 in the subclass $\mathcal{H}_\sigma(\gamma,1)$, we obtain (3.21). Thus, the proof is complete.

Theorem 3.5

If $f \in \mathcal{H}_\sigma(\gamma,\beta)$, $0 \le \gamma < 1$ and $\beta > 0$, then

$$|H_2(2)| \le \frac{(1-\gamma)^{3/2}}{(1+3\beta)\sqrt{6(1+2\beta)}} + \left(\frac{1-\gamma}{1+\beta}\right)^4 + \frac{4(1-\gamma)^3}{3(1+2\beta)(1+\beta)^2} + \frac{4}{9}\left(\frac{1-\gamma}{1+2\beta}\right)^2 \tag{3.22}$$

Proof

Since

$$|H_2(2)| \le |a_2||a_4| + |a_3^2|$$

Hence, according to theorem 3.2 where

$$|a_4| \le \frac{(1-\gamma)}{2(1+3\beta)}$$

and theorem 2.10 where

$$|a_2| \le \sqrt{\frac{2(1-\gamma)}{3(1+2\beta)}} \quad , \quad |a_3| \le \frac{(1-\gamma)^2}{(1+\beta)^2} + \frac{2(1-\gamma)}{3(1+2\beta)}$$

we obtain (3.22). Thus, the proof is complete.

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4. CONCLUSION

In this paper, we obtained an estimate for the second Hankel determinant in the two subclasses $\mathscr{M}_\sigma(\beta,\lambda)$ and $\mathcal{H}_\sigma(\gamma,\beta)$. We also obtained an estimate for the fourth coefficient in the subclass $\mathcal{H}_\sigma(\gamma,\beta)$. This, in turn, led to obtaining an estimate for both the fourth coefficient and the second Hankel determinant in the subclass $\mathcal{H}_\sigma(\gamma,1)$.

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